UNIVERSITY OF BIRMINGHAM | COLLEGE OF SOCIAL SCIENCES

Arbitrary and Necessary – *A way of viewing the mathematics curriculum*

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Arbitrary and Necessary – *A way of viewing the mathematics curriculum*

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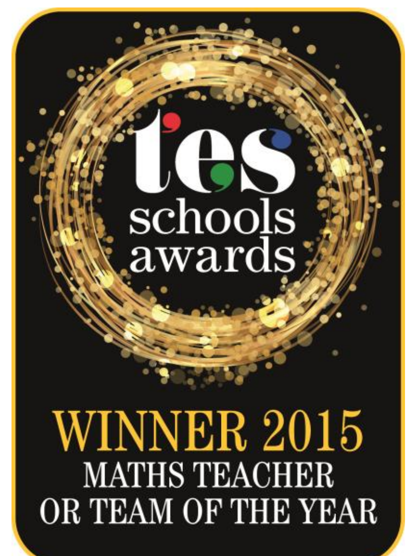
There are many factors to consider when teaching all-attainment groups. Dave Hewitt's (1999; 2001a; b) way of dividing the curriculum into *arbitrary* and *necessary* has been a helpful distinction for me when working on mathematics with learners. In this active session I will outline the theoretical perspectives and work on some tasks to illustrate the mathematics teacher's role in both assisting memory and educating awareness. This may be useful in terms of both the activities offered and when planning to meet the needs of all mathematics learners.

Tom is a mathematics teacher and Lecturer in Secondary Mathematics at the University of Birmingham.

- Teach on PGDipEd at University of Birmingham
- Previously head of maths at KNGS
- Mixed attainment groups
- Best Practice in Grouping



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“‘*Obvious*’ is the most dangerous word in mathematics”

Eric Temple Bell

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Arbitrary and Necessary: a way of viewing the mathematics curriculum

Dave Hewitt:

If I'm having to remember…

… then I'm not working on mathematics.

Hewitt, D. (1999) 'Arbitrary and necessary part 1: A way of viewing the mathematics curriculum', *For the Learning of Mathematics, 19(3), pp. 2-9.*

How would you answer this question?

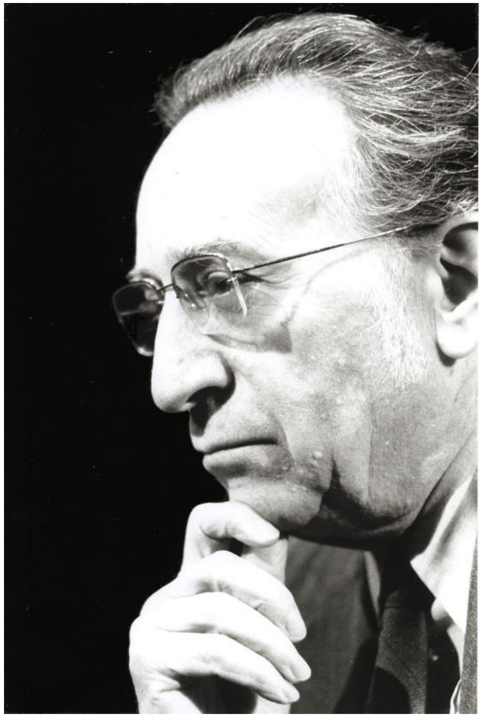
Student: *What is the name of a four-sided shape which has all four angles equal to 90 degrees and all sides the same length?*

Teacher: *A square.*

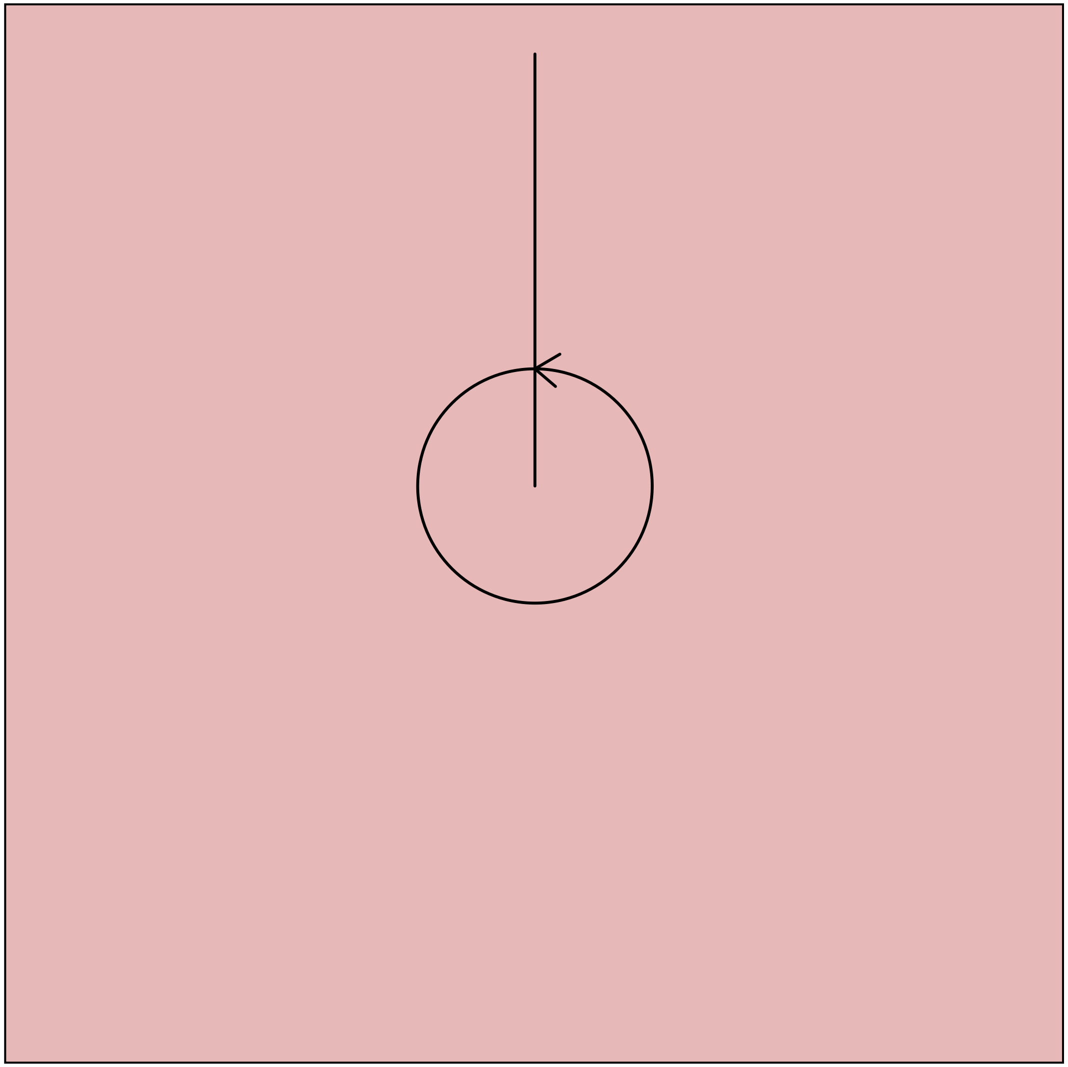
Student: *Why?*

Caleb Gattegno

… there is knowledge that is distinguished sharply from awareness - the knowledge solely entrusted to one's memory, such as the label for such an object or a telephone number, items which are arbitrary. Without someone else, that knowledge would not exist for us (p55).



Gattegno, C. (1987), *The Science of Education. Part 1 - Theoretical Considerations*, New York: Educational Solutions.



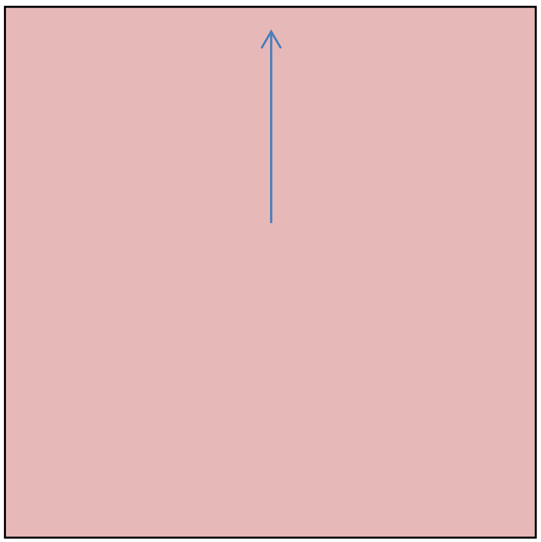
Arbitrary & Necessary

- Names and conventions
- Cannot be worked out
- Might be so

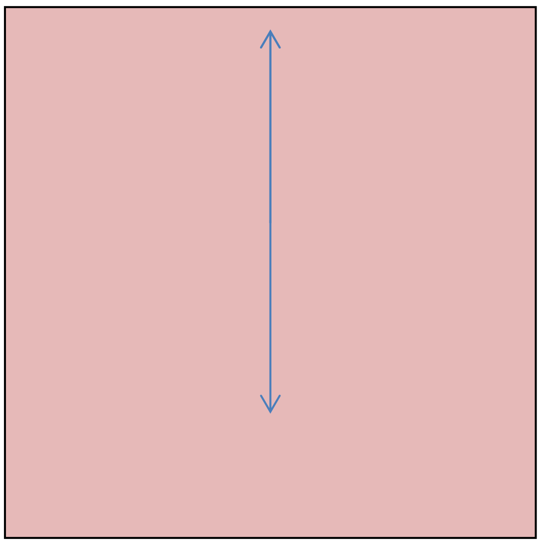
- Properties and relationships
- Can be worked out
- Must be so

Arbitrary & Necessary

- A full turn is 360°

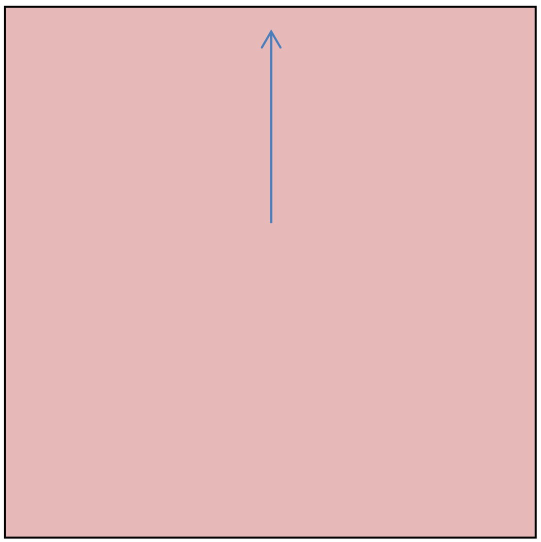


- A half turn is 180°

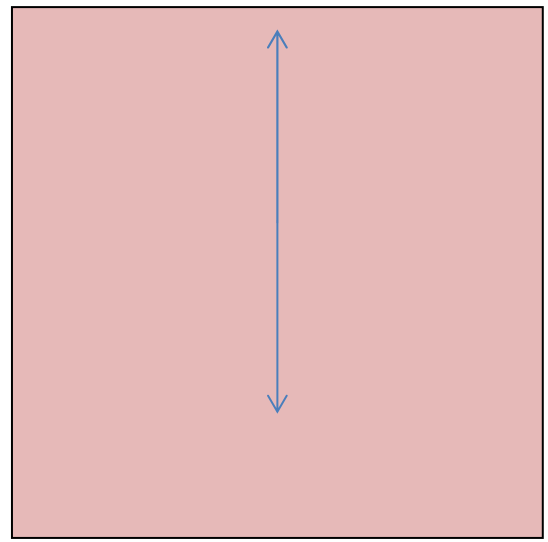


Arbitrary & Necessary

- A full turn is 2π

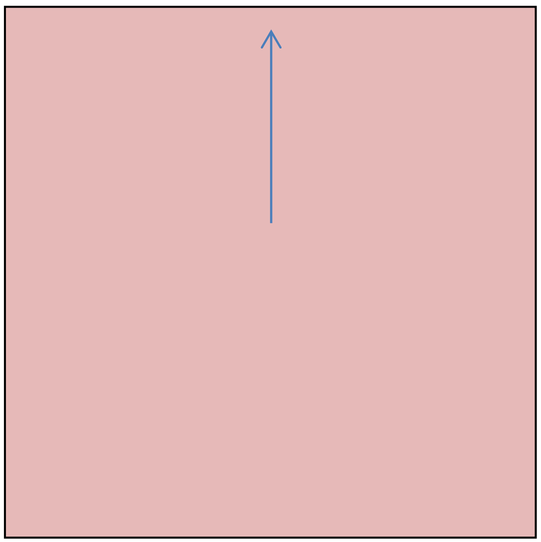


- A half turn is π

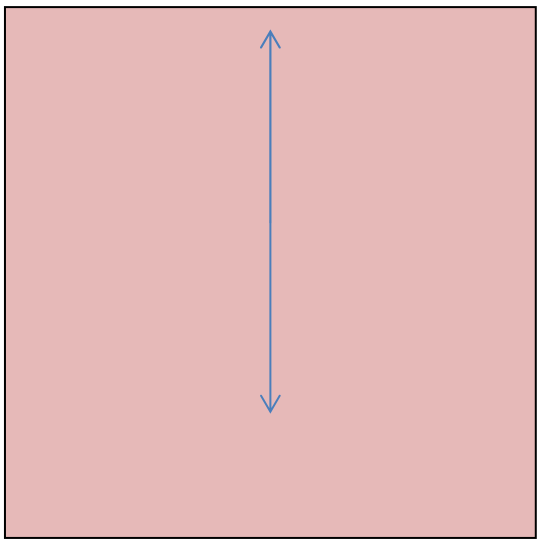


Arbitrary & Necessary

- A full turn is 400

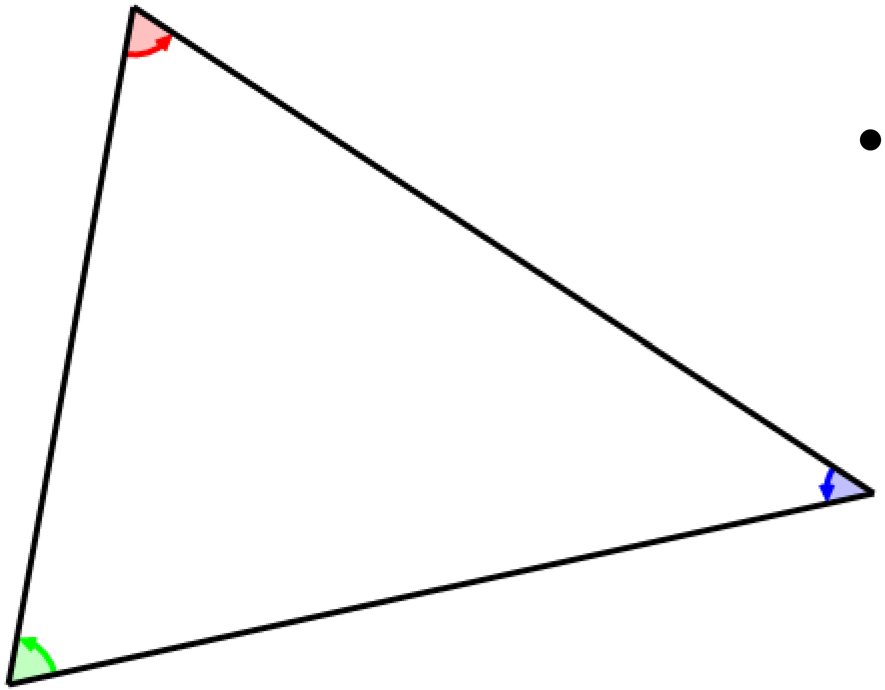


- A half turn is 200



Arbitrary & Necessary

- Shape is called a triangle



- Angle sum is the same for every triangle
- Angles sum to half a turn

Arbitrary & Necessary

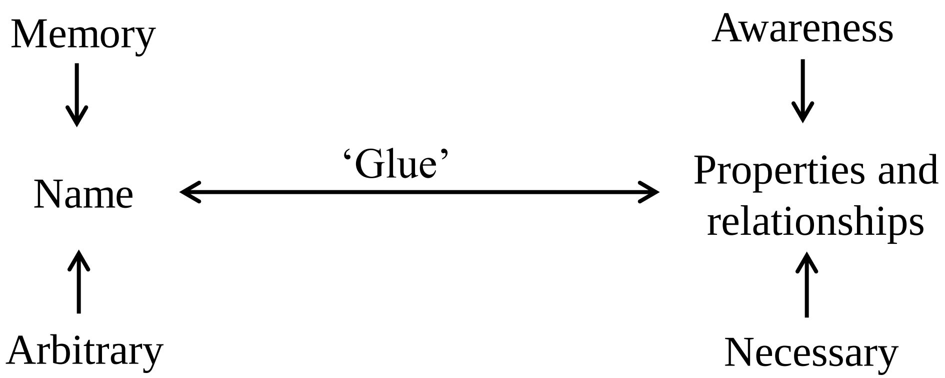
- 1, 2, 3, 4, 5, 6, 7, …
- 4, 1, 8, 7, 2, 9, 3, …

- 2 + 3 = 5
- 1 + 8 = 2

Memory

Name ‘Glue’ Properties and relationships

Arbitrary


Memory
Name
Arbitrary
'Glue'
Awareness
Properties and relationships
Necessary

Arbitrary	students <u>need</u> to be informed of the arbitrary by someone else	*Realm of Memory*
Necessary	students <u>can</u> become aware of what is necessary without being informed of it by someone else	*Realm of Awareness*

Think of some examples in both categories

Cannot be worked out (<u>might</u> be so)	Can be worked out (<u>must</u> be so)

Cannot be worked out **(<u>might</u> be so)**	**Can be worked out** **(<u>must</u> be so)**
• Names of shapes • Definitions of ...... • Measuring bearings from north • x and y co-ordinates • How heavy is a kg? • How long is a metre? • Terminology - e.g. names of theorems, such as ‘factor’ theorem • word/label	• Interior angles of regular polygons • $V = IR \rightarrow I = V/R$ • Solution of a linear equation • What happens to a number if multiplied by <1 or >1 • Rough estimates of measurements • 2×3 • Finding factors of $a^3 + b^3$ • Finding angles or lengths in triangle problems, for example based upon this triangle: • Property of primeness • Symmetry
Summarised as: words, symbols, notation and conventions.	Summarised as: properties and relationships.

	Student	Teacher	Mode of teaching
Arbitrary	All students need to be informed of the arbitrary by someone else	A teacher needs to inform students of the arbitrary	*Assisting Memory*
Necessary	Some students can become aware of what is necessary without being informed of it by someone else	A teacher does not need to inform students of what is necessary. Instead a teacher might use questions or provide activities	*Educating Awareness*

received wisdom

In a lesson I observed, some 14-15 year olds were working on solving simultaneous equations, and one male student was having difficulties with re-arranging an equation. He had written:

$$\begin{aligned} x - y &= 2 \\ y &= 2 - x \end{aligned}$$

I asked him about the '-' sign in front of the y and his response was to re-write the second equation to:

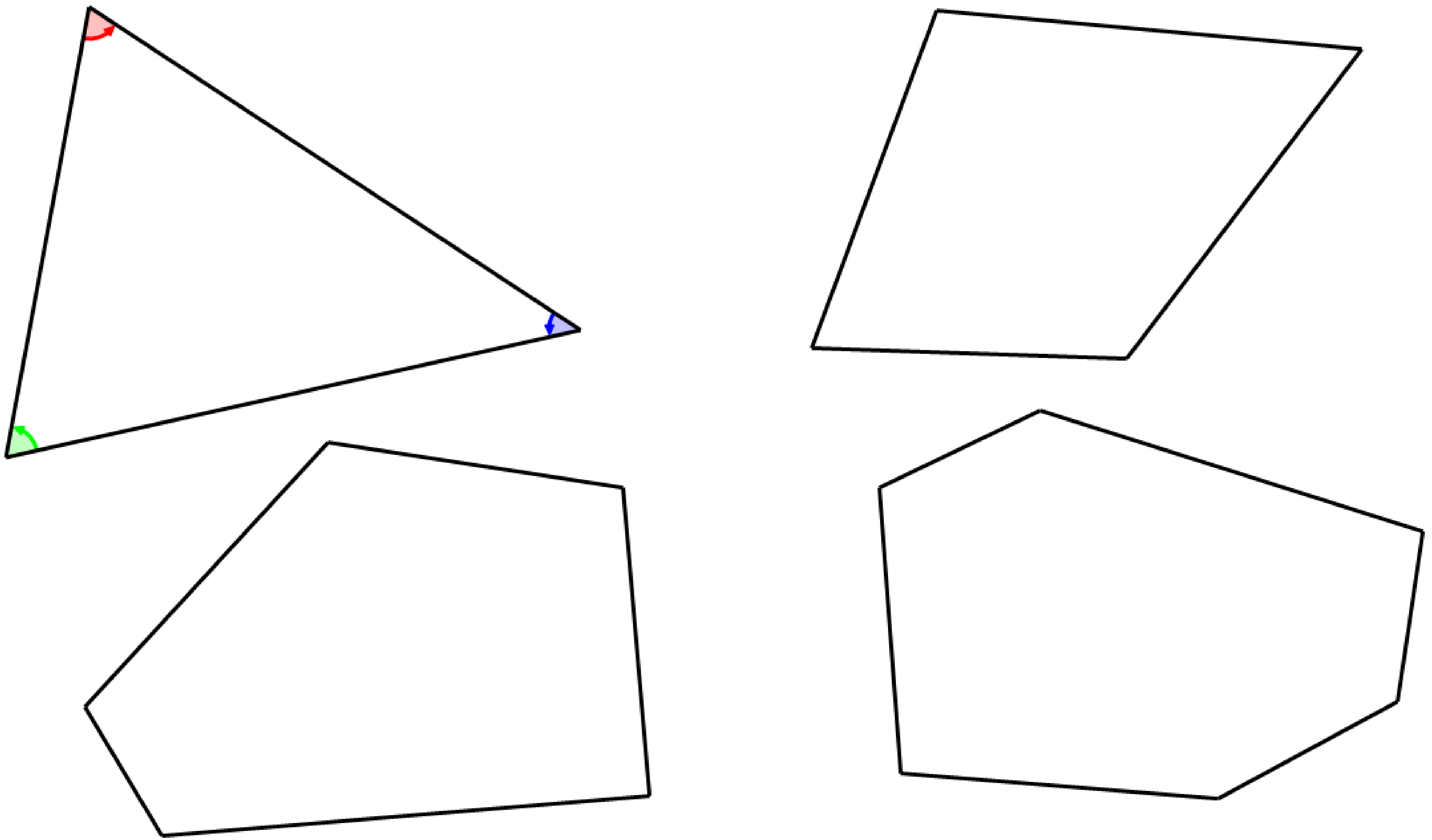
$$y = 2 + x$$

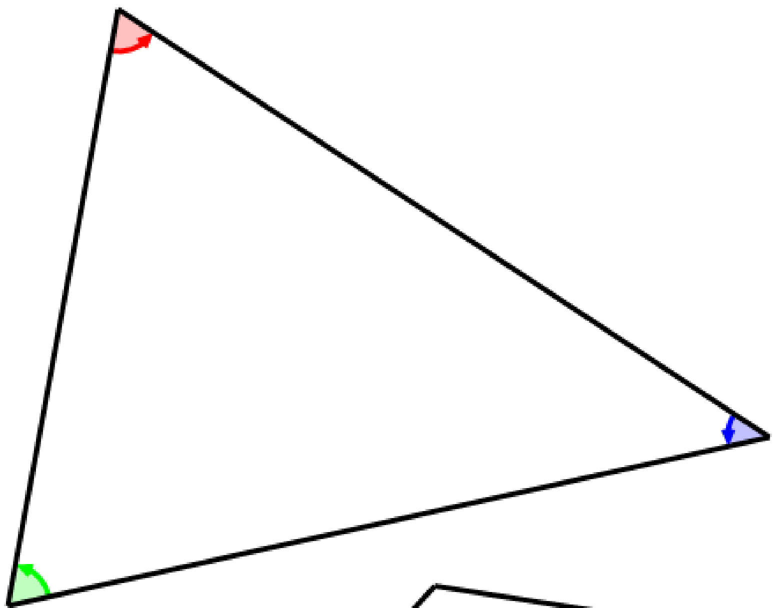
I said that I felt he had done the correct thing when taking away the x but that there was still a '-' sign in front of the y. I wrote a '-' in front of the y in the original second equation:

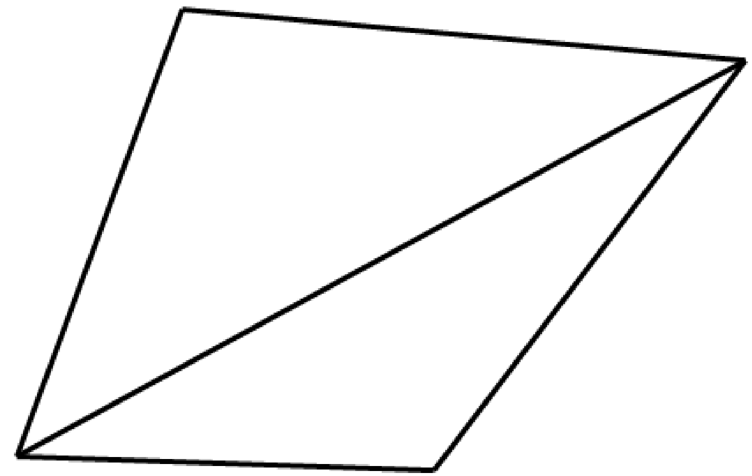
$$-y = 2 - x$$

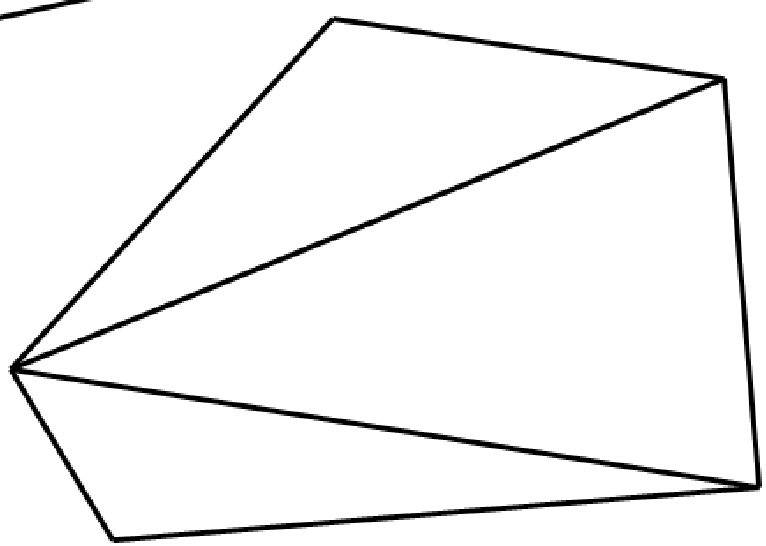
He then changed both the subtractions to additions saying *two negatives make a positive*:

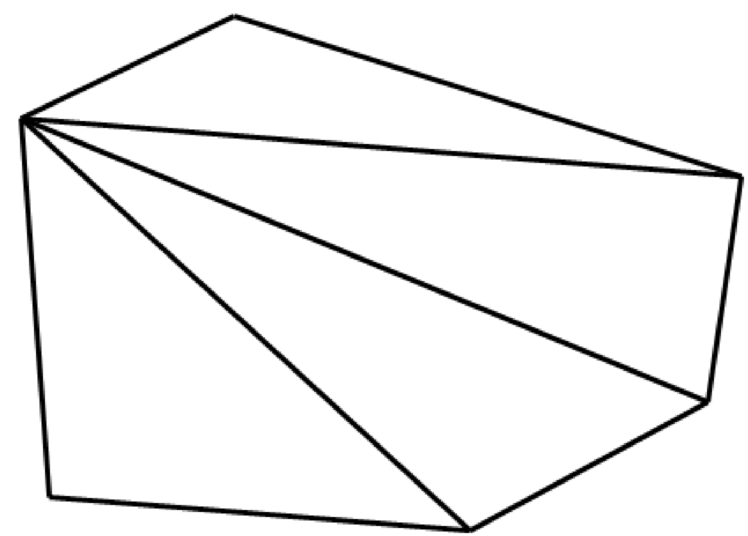
$$+y = 2 + x$$











	ARBITRARY		NECESSARY	
TEACHER	teacher - informs	teacher - does not inform	teacher - informs	teacher - gives appropriate activity
STUDENT	students - have to memorise	students - have to invent	*Received wisdom* students - have to memorise unless they succeed in using their awareness to come to know	students - use awareness to come to know

The Arbitrary:

Assisting Memory

What was the number I asked you to remember?

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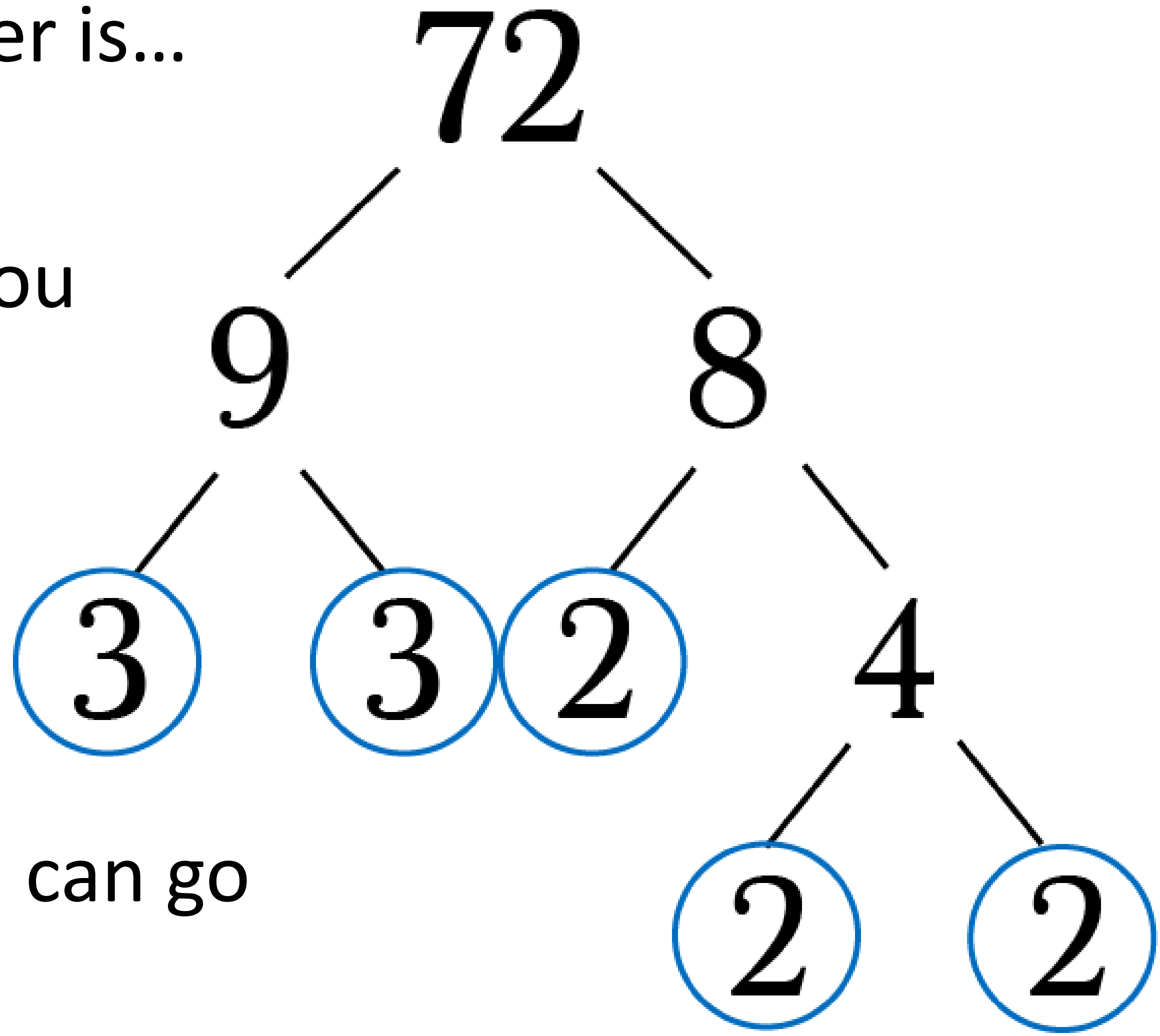
Introducing the arbitrary

- A prime number is…
- 2,3,5,7,11,…
- … stop when you reach a prime



- Or…
- Stop when you can go no further
- These numbers are *prime*

Commutative

An operation, $\oplus$, on a set S, is *commutative* if

$$x \oplus y = y \oplus x$$

for all x and y in S.

You have only one of each of the following numbers:

1, 2 and 3

Using only two of these and an addition sign…

… make as many different answers as you can.

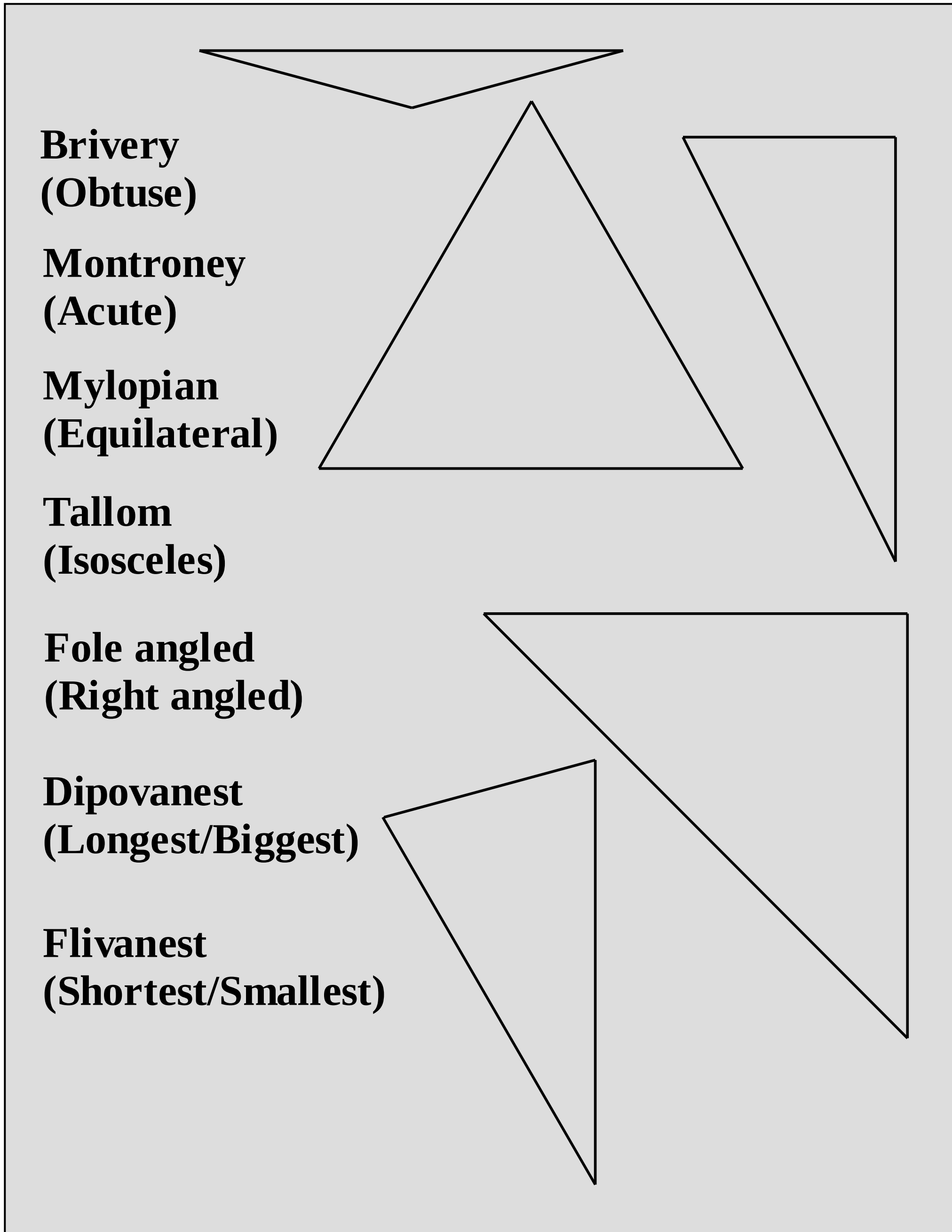
You have only one of each of the following numbers:

1, 2 and 3

Using only two of these and a subtraction sign…

… make as many different answers as you can.

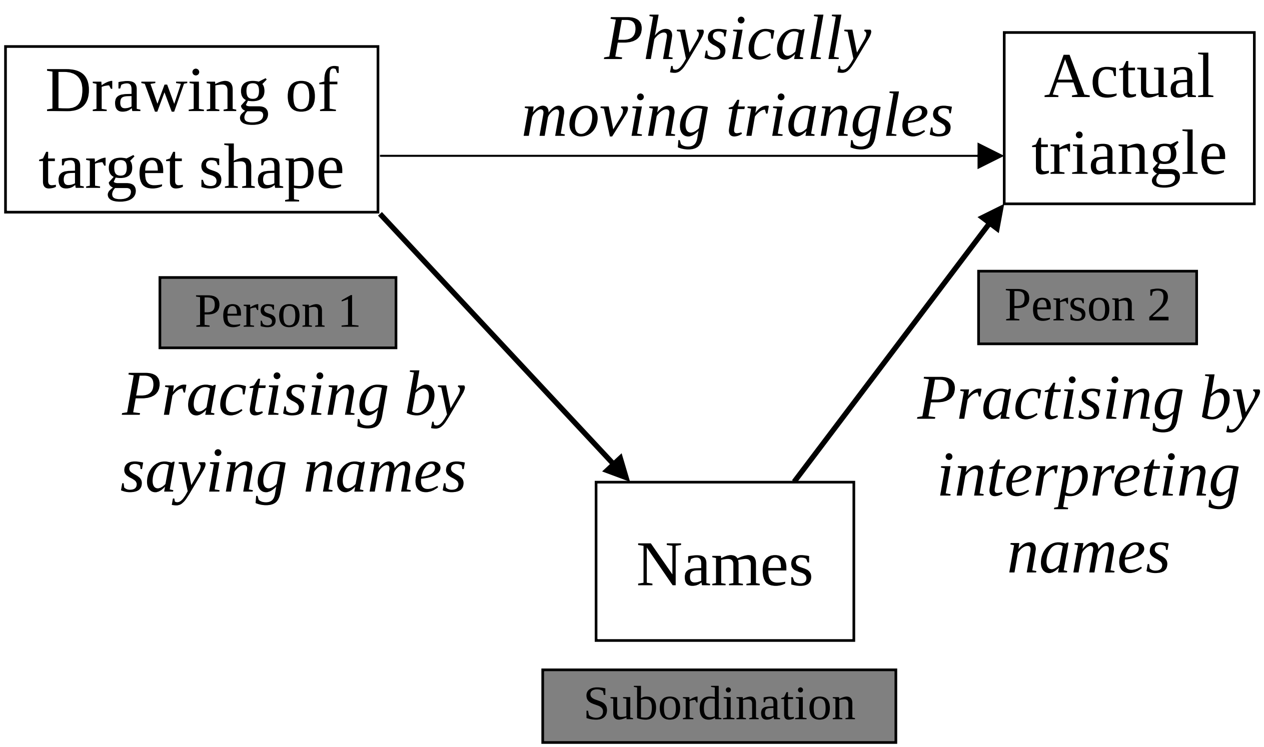
Practising the arbitrary



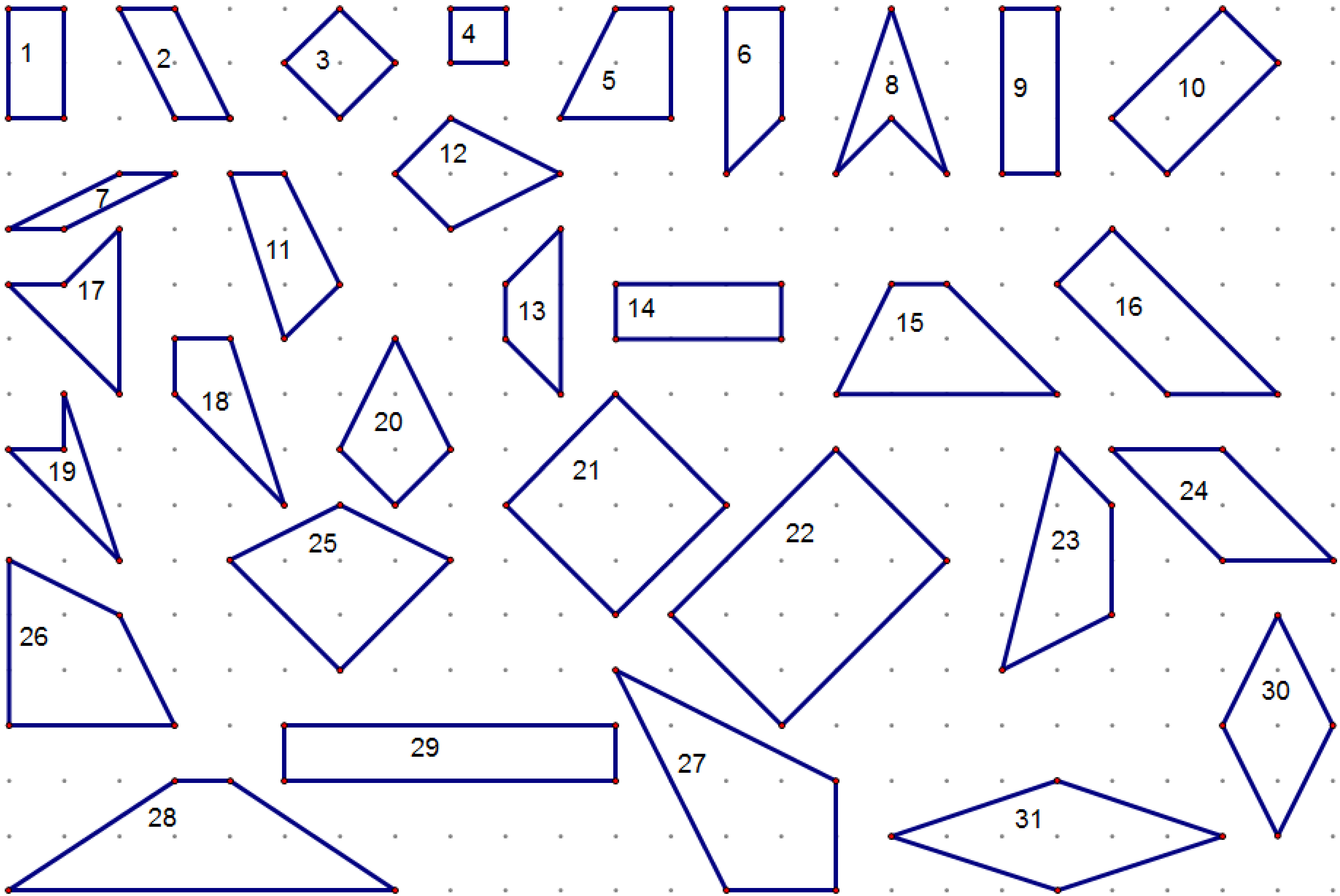
<u>Rules</u>:

No pointing by either person,
either directly (pointing)
or indirectly
("the one next to you", "not that one")

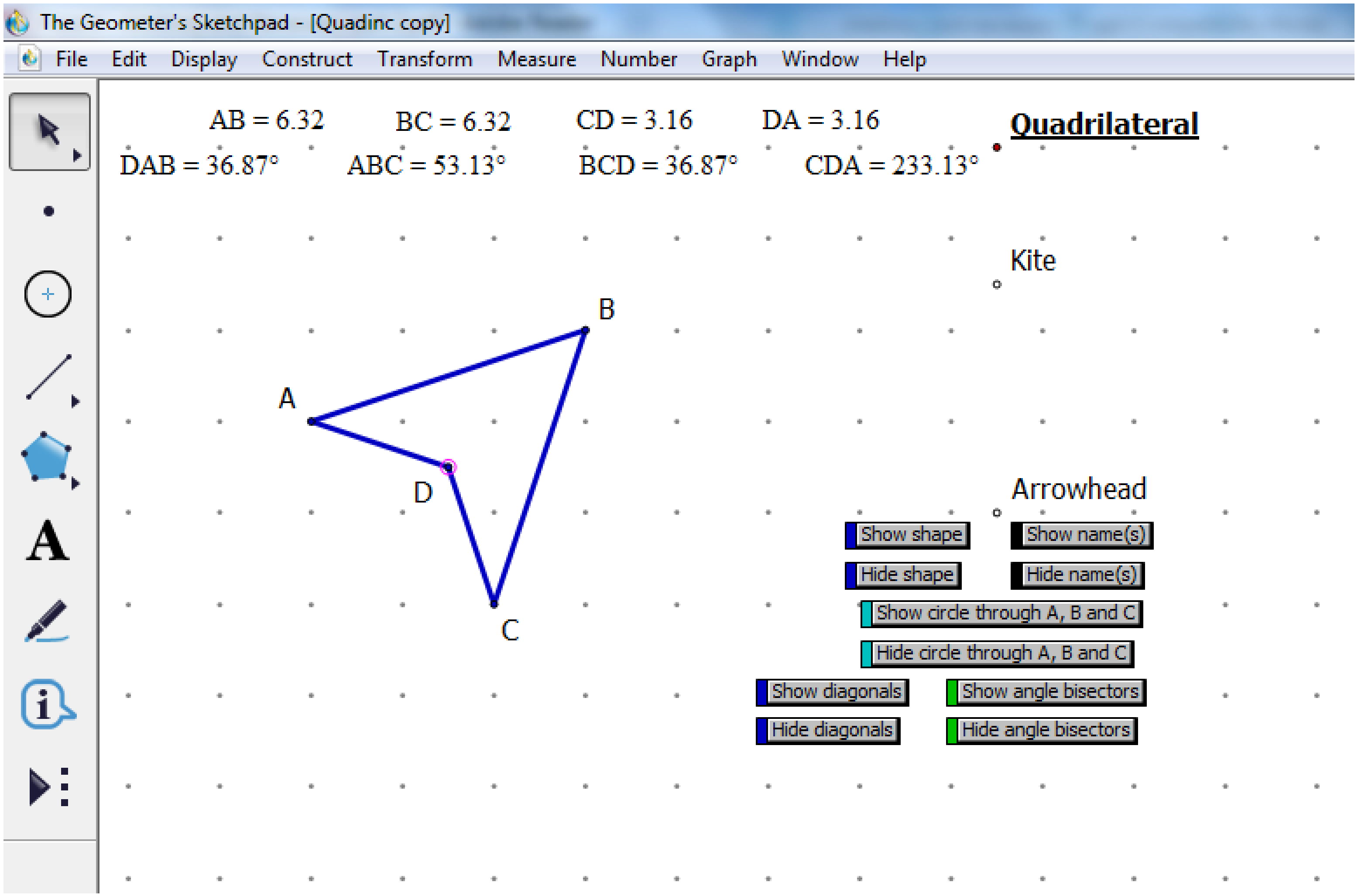
Person moving shapes cannot speak


Drawing of target shape
Physically moving triangles
Actual triangle
Person 1
Practising by saying names
Person 2
Practising by interpreting names
Names
Subordination

Stressing names and properties together



Stressing names and properties together



Quadinc - Available from: https://www.atm.org.uk/Shop/Active-Geometry-for-Geometre-and-Sketchpad-incl-demos-CD/sof063

The Necessary:

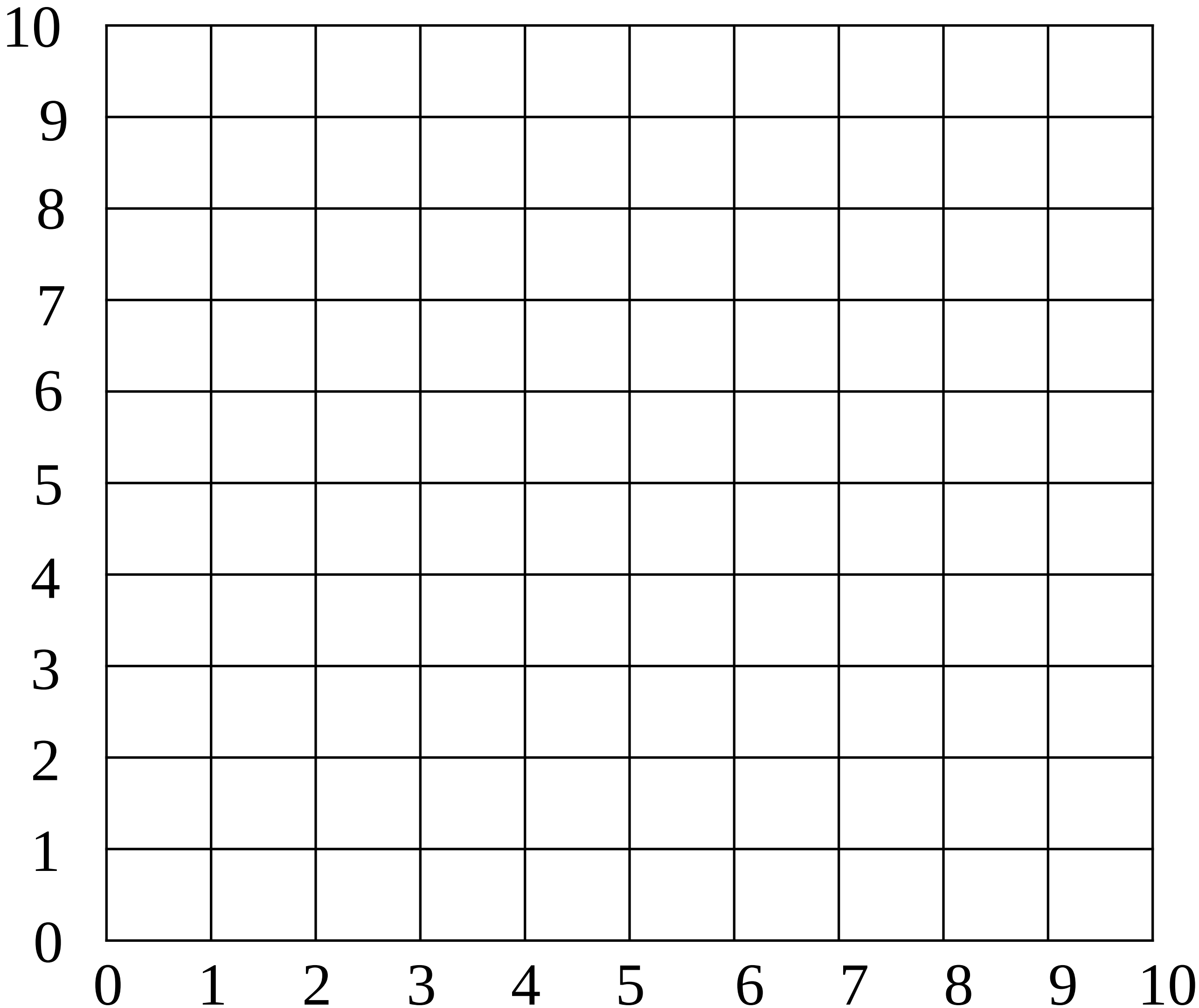
Educating Awareness

Educating awareness
(of the necessary)…

… whilst assisting memory
(of the arbitrary)

https://www.desmos.com/calculator

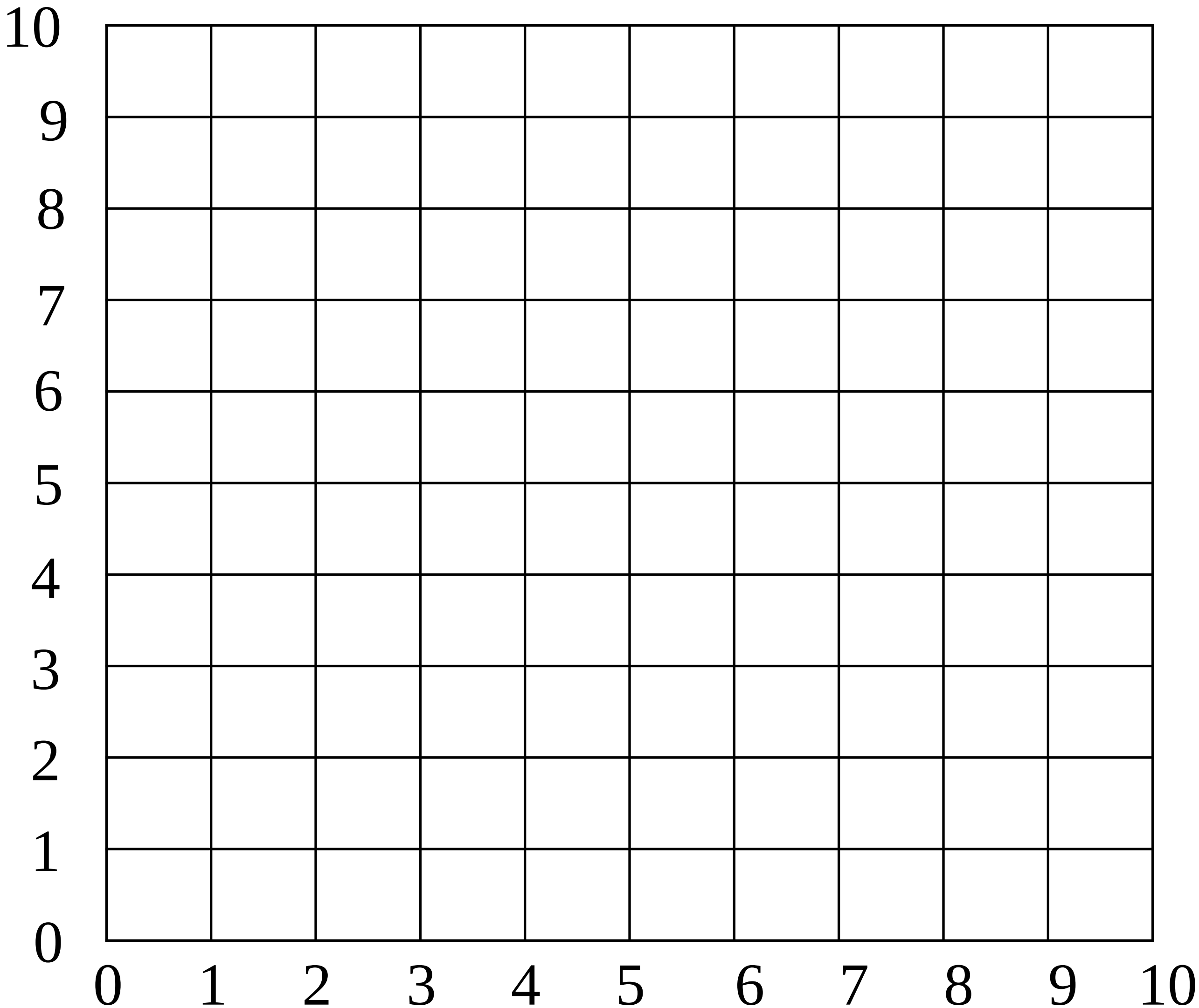
“Make some squares” game



Practising Mathematics - Available from https://www.atm.org.uk/Shop/Practising-Mathematics---Developing-the-Mathematician-as-Well-as-the-Mathematics-Book-and-Download/ACT107pk

Do we meet vector TF

"Do we meet?" game



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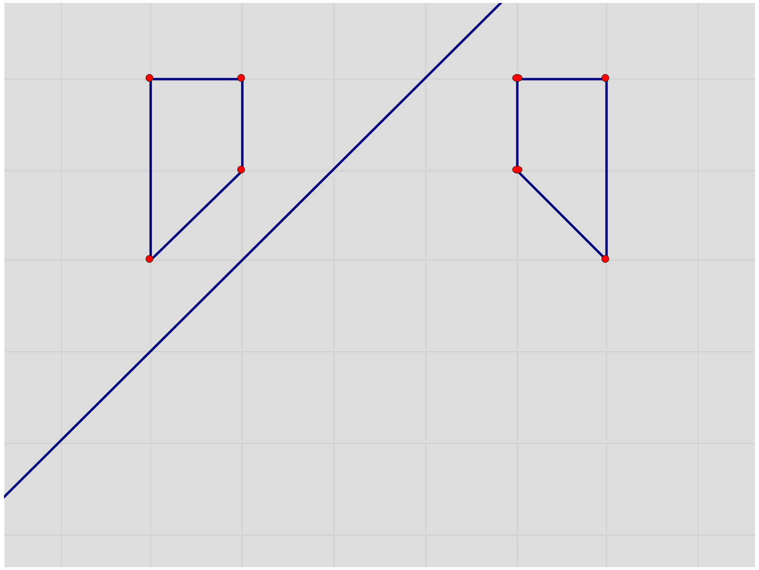
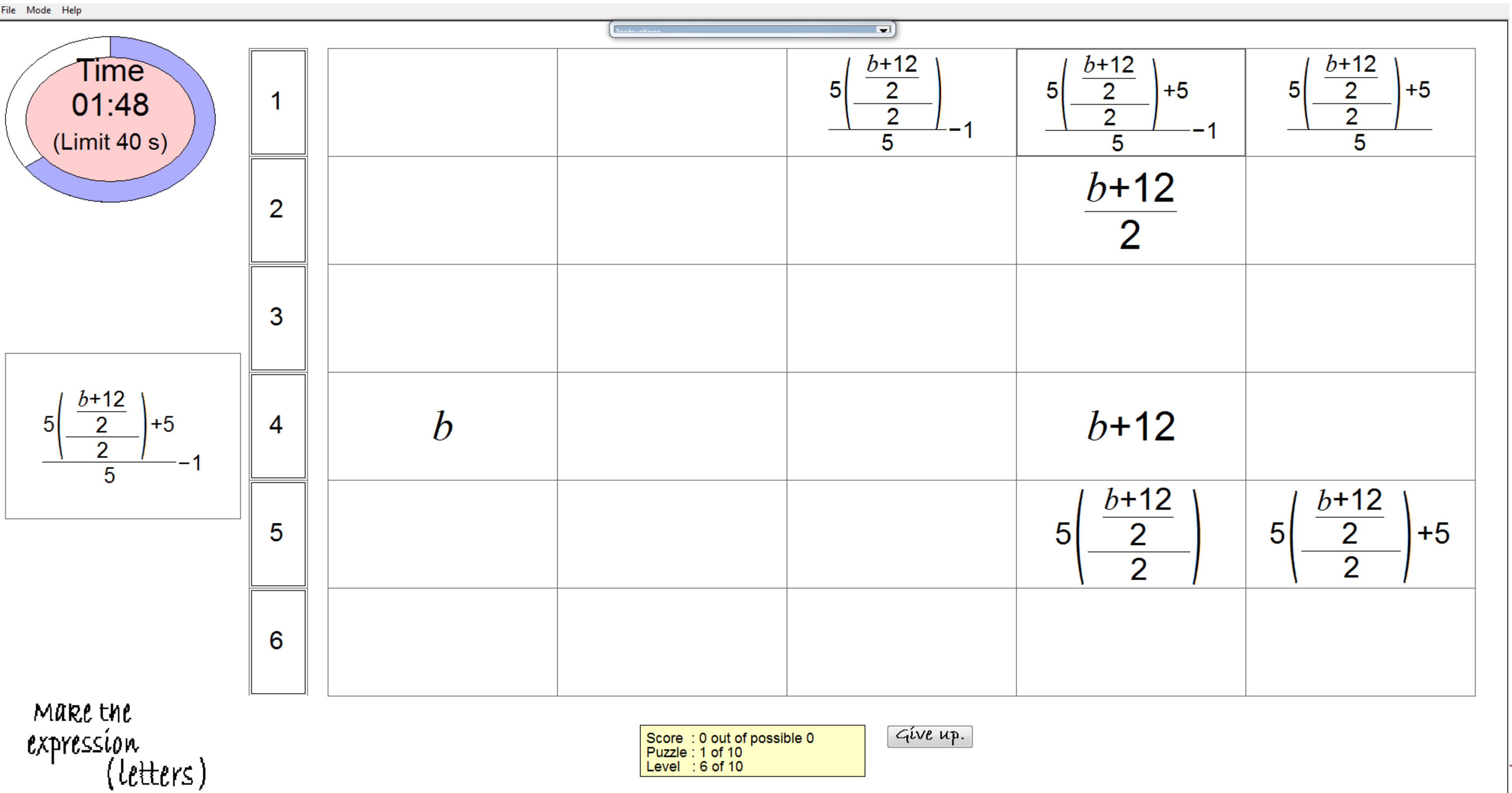
Opportunities:
what awareness am I trying to educate?

Listening and watching students – what might they be thinking, to do what they are doing?



Feedback:
seeing the consequences of actions



Grify Algebra- Available from https://www.atm.org.uk/Shop/Primary-Education/Software-Media/Grid-Algebra---Site-Licence/sof074 https://nrich.maths.org/5460

Feedback:
seeing the consequences of actions

Feedback:
seeing the consequences of actions

Using Logo to try to draw a square:

Forward 100

Right turn 90

Forward 100

Right turn 90

Forward 100

Right turn 90

Forward 100

Right turn 90

Feedback: seeing the consequences of actions

Using Logo to try to draw an equilateral triangle:

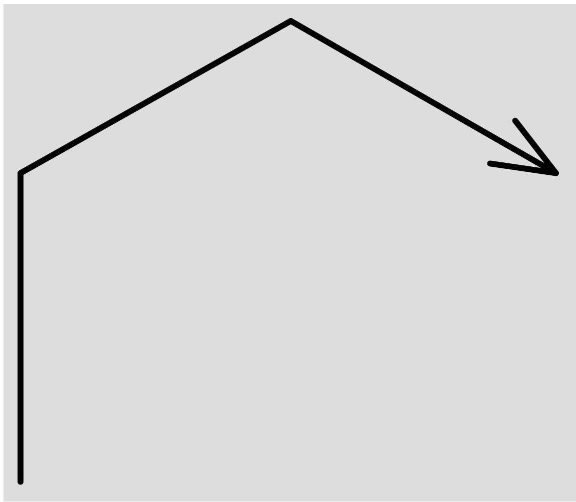
Forward 100

Right turn 60

forward 100

Right turn 60

Forward 100



Directing attention

Year 8 student was trying to solve:

$$7x + 2 = 5 - 3x$$

He had got to the following and was stuck:

$$10x + 2 = 5$$

I covered up the $10x$ and asked what number was under my finger.

Accounting for what is noticed

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36
37	38	39	40	41	42
43	44	45	46	47	48
49	50	51	52	53	54
55	56	57	58	59	60
61	62	63	64	65	66
67	68	69	70	71	72
73	74	75	76	77	78
79	80	81	82	83	84
85	86	87	88	89	90

Accounting for what is noticed

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36
37	38	39	40	41	42
43	44	45	46	47	48
49	50	51	52	53	54
55	56	57	58	59	60
61	62	63	64	65	66
67	68	69	70	71	72
73	74	75	76	77	78
79	80	81	82	83	84
85	86	87	88	89	90

Forcing awareness:
a sequence of questions

$$\frac{3}{7} \div \frac{5}{12} = \frac{3}{7} \times \frac{12}{5}$$

How many halves are there in one?
How many quarters are there in one?
How many tenths are there in one?
How many thirds are there in one?
...
How many twenty-fourths are there in one?
How many two-hundred and forty-ninths are there in one?
How many five million, eight hundred and sixths are there in one?
...
How many flinkerty-flooths are there in one?
How many xths are there in one?
*** $1 \div n = \frac{1}{n}$ ***

How many halves are there in one?
How many halves are there in two?
How many halves are there in five?
...
How many halves are there in nine hundred and fifty-two?
How many halves are there in twenty trillion, five million and seven?
How many halves are there in flinkerty-floo?
How many halves are there in x?
*** $x \div \frac{1}{2} = x \times 2$ ***

How many quarters are there in one?
How many quarters are there in two?
How many quarters are there in eight?
...
How many quarters are there in twenty-seven?
...
How many tenths are there in one?
How many tenths are there in fourteen?
How many tenths are there in two thousand and fifty-one?
...
How many sixteenths are there in one?
How many sixteenths are there in six?
...

How many four hundred and twentieths are there in one?
How many four hundred and twentieths are there in nine?
...
How many flinkerty-flooths are there in one?
How many flinkerty-flooths are there in ten?
How many flinkerty-flooths are there in seven thousand?
How many flinkerty-flooths are there in nine million, two hundred and eight?
How many flinkerty-flooths are there in zipperly-bond?
How many flinkerty-flooths are there in x?
How many nths are there in x?
*** $x \div \frac{1}{n} = x \times n$ ***

How many times does one quarter go into one?
How many times does one quarter go into seventeen?
How many times does two quarters go into one?
How many times does two quarters go into three?
How many times does two quarters go into twenty-five?
...
How many times does two quarters go into three?
How many times does four quarters go into three?
How many times does six quarters go into three?
How many times does five quarters go into three?
How many times does seventeen quarters go into three?
...
How many times does six quarters go into three?
How many times does six quarters go into nine?
How many times does six quarters go into forty-one?
...
How many times does six quarters go into three?
How many times does six quarters go into thirty-seven?
How many times does nine quarters go into thirty-seven?
How many times does two hundred and seven quarters go into thirty-seven?
How many times does two hundred and seven quarters go into nine million and six?
...

How many times does one tenth go into one?
How many times does one tenth go into twenty?
How many times does two tenths go into twenty?
How many times does five tenths go into twenty?
How many times does sixty-three tenths go into twenty?
How many times does sixty-three tenths go into ninety-four?
...
How many times does one flinkerty-flooth go into one?
How many times does one nth go into one?
How many times does one nth go into seventy five?
How many times does one nth go into x?
How many times does two nths go into x?
How many times does sixty-one nths go into x?
How many times does p nths go into x?
*** $x \div \frac{p}{n} = x \times \frac{n}{p}$ ***

How many times does p nths go into x?
How many times does p nths go into four?
How many times does p nths go into eighty-seven?
How many times does p nths go into sixty-nine?
How many times does p nths go into zipperly-bond?
How many times does p nths go into two quarters?
How many times does p nths go into seven twelfths?
...
How many times does p nths go into x yths?
*** $\frac{x}{y} \div \frac{p}{n} = \frac{x}{y} \times \frac{n}{p}$ ***

Hewitt, D. (2001) 'Arbitrary and necessary: Part 3 educating awareness', *For the Learning of Mathematics, 21(2), pp. 37-49.*

Arbitrary and necessary (Hewitt 1999, 2001a, b) divides the mathematics curriculum into those things which are social labels and conventions (the arbitrary) and those which concern properties and relationships (the necessary). The first group of socially agreed labels and conventions include names such as *square*, *five* and *horizontal*. These words have become established over a long period of time and, although there may have been good reasons at the time why each word was adopted, there is nothing necessary about the choice of each particular word. Indeed, in another language, different words are used. So there is nothing about a square which means it *has* to be called *square*.

Likewise with conventions that the first number of (2, 3) is the x co-ordinate and not the y co-ordinate, or that a whole turn is divided up into 360 when deciding a way to measure turn. There are historical reasons why 360 was chosen that relate to the culture at the time (the Babylonians used a base 60 number system) and mathematical convenience (360 has many factors), but that still does not change the fact that there was choice and it did not *have* to be 360. As such, for a learner today, these conventions can feel arbitrary. I argue that the focus on the arbitrary for a learner should be one of accepting and adopting, rather than questioning and challenging. Whilst acknowledging that it can be of considerable interest to explore the historical stories of why certain names and conventions were adopted, the focus for a mathematics curriculum is on the adoption and use of the arbitrary in order to work on what is necessary.

Those things which are necessary *have* to be how they are, given certain arbitrary decisions. There is no choice. So, if we are working with the arbitrary choice of 360° in a whole turn, then it is necessary that half a turn will have 180° and reasons can be given for why this must be so. The mathematics lies within the relationship between a whole turn and half a turn rather than the particular numbers involved. Indeed, a similar relationship must occur when talking about radians as much as degrees. The necessary concerns properties and relationships and this is where the mathematics lies.

The equation $2(r + 3) = 2r + 6$ involves many arbitrary labels and conventions in terms of the symbolism and notation used. However, the relationship is about operations: that if I add three and then double, it gives me the same numerical result as doubling and then adding six. What number I started with is irreverent. This remains true no matter how I might express it notationally. So, suppose a learner were to work on a problem where the task is to find a general rule, such as $2(r + 3)$. However, instead of writing this, they write $r + 3 \times 2$. What they have written is ‘wrong’, but what is of importance is to find out *what they are wrong about*. If they think that the rule is that you take r and add to it 3×2, then they are wrong mathematically, but right notationally. If they think the rule is r add three and then multiply by two, then they are right mathematically, but wrong notationally. The difference is significant for a teacher. In the first case, the leaner’s awareness of the mathematical situation needs to be addressed; whereas, in the second case, it is only a matter of helping them adopt a convention about how the rule is written down. The first concerns the necessary and the second, the arbitrary.

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